An Introduction to Low Density Parity Check (LDPC) Codes

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Outline

1. History of LDPC codes
2. Properties of LDPC codes
3. Basics of LDPC codes
   - Encoding of LDPC codes
   - Iterative decoding of LDPC codes
   - Simplified approximations of LDPC decoders
4. Applications of LDPC codes

Features of LDPC Codes

- Approaching Shannon capacity
  - For example, 0.3 dB from Shannon limit
  - Irregular LDPC code with code length 1 million. (Richardson:1999)
  - An closer design from (Chung:2001), 0.0045 dB away from capapcity
- Good block error correcting performance
- Low error floor
  - The minimum distance is proportional to code length
- Linear decoding complexity in time
- Suitable for parallel implementation

History of LDPC Codes

- Invented by Robert Gallager in his 1960 MIT Ph. D. dissertation. Long time being ignored due to
  1. Requirement of high complexity computation
  2. Introduction of Reed-Solomon codes
  3. The concatenated RS and convolutional codes were considered perfectly suitable for error control coding.
- Rediscovered by MacKay(1999) and Richardson/Urbanke(1998).

Foundamentals of Linear Block Codes

- The structure of a code is completely described by the generator matrix $\mathbf{G}$ or the parity check matrix $\mathbf{H}$.
- The capacity of correcting symbol errors in a codeword is determined by the minimum distance (d_{min}).
  - d_{min} is the least weight of the rows in $\mathbf{G}$.
  - d_{min} is the least number of columns in $\mathbf{H}$ that sum up to $\mathbf{0}$.
  - Example: (7, 4) Hamming code

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Properties of LDPC Codes

- **H** is sparse.
  - Very few 1's in each row and column.
  - Expected large minimum distance.
- Regular LDPC codes
  - **H** contains exactly W_c 1's per column and exactly $W_r = W_c(n/m)$ 1's per row, where $W_c << m$.
  - The above definition implies that $W_r << n$.
  - $W_c \geq 3$ is necessary for good codes.
- If the number of 1's per column or row is not constant, the code is an *irregular* LDPC code.
  - Usually irregular LDPC codes outperforms regular LDPC codes.

A Sample LDPC Code

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 1 & \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 1 & \\ 0 & 0 & 1 & 0 & \\ & & \vdots & & \end{bmatrix}$$

- $W_c = 3$;
- Any two columns have an overlap of at most one 1;
- The sparse property allows us to avoid overlapping;
- In the part of $\mathbf{H}$ shown, there does not exist a set of columns that add up to $\mathbf{0}$.
- The above facts make the d_{min} large;
- $\mathbf{G}$ is found by Gaussian elimination.
  - $\mathbf{H}$ can be put in the form $\mathbf{H} = \left[\mathbf{P}^T \vdots \mathbf{I} \right]$.
  - The generator matrix $\mathbf{G} = \left[\mathbf{I} \vdots \mathbf{P} \right]$.

Encoding of LDPC Codes

- General encoding of systematic linear block codes

$$\mathbf{c} = \mathbf{x}\mathbf{G} = \left[\mathbf{x} \vdots \mathbf{x}\mathbf{P} \right] \tag{1}$$

- Issues with LDPC codes
  - The size of $\mathbf{G}$ is very large.
  - $\mathbf{G}$ is not generally sparse.
  - Example: A (10000, 5000) LDPC code.
    * $\mathbf{P}$ is 5000×5000.
    * We may assume that the density of 1's in $\mathbf{P}$ is 0.5
    * There are 12.5×10^6 1's in $\mathbf{P}$
    * 12.5×10^6 addition (XOR) operations are required to encode one codeword.
- An alternative approach to simplified encoding is to design the LDPC code via algebraic or geometric methods.
  - Such "structured" codes can be encoded with shift register circuits.

Iterative Decoding of LDPC Codes

- General decoding of linear block codes
  - Only if $\mathbf{c}$ is a valid codeword, we have

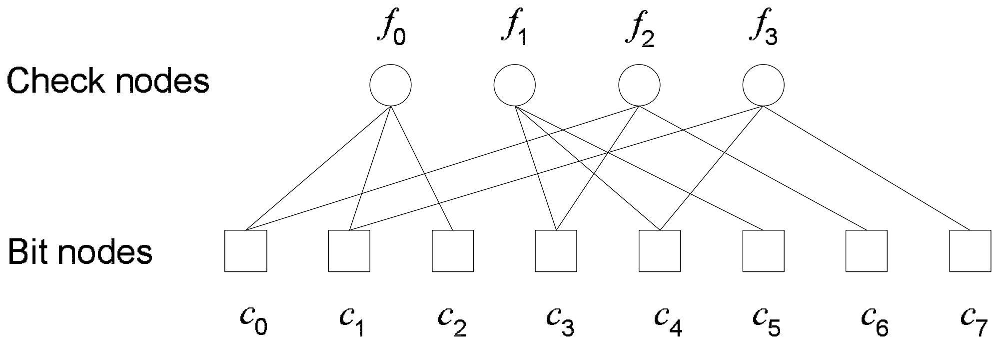
$$\mathbf{c}\mathbf{H}^T = \mathbf{0} \tag{2}$$

  - For binary symmetric channel (BSC), the received codeword is $\mathbf{c}$ added with an error vector $\mathbf{e}$.
  - The decoder needs to find out $\mathbf{e}$ and flip the corresponding bits.
  - The decoding algorithm is based on linear algebra.
- Graph-based algorithms
  - Sum-product algorithm for general graph-based codes;
  - MAP (BCJR) algorithm for trellis graph-based codes;
  - Message passing algorithm for bipartite graph-based codes.

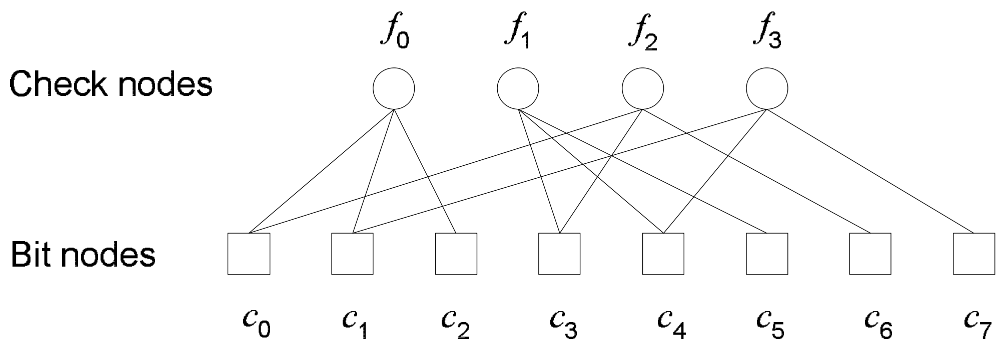
Tanner Graph

- Bipartite graph
  A bipartite graph is an undirected graph whose nodes may be separeted into two classes, where edges only connect two nodes not residing in the same class.
- Tanner graph
  The two classes of nodes in a Tanner graph are the *bit nodes* and the *check nodes*.
- Example: An (8, 4) product code

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



Message Passing Algorithm (1/4)



- Maximum likelihood (ML) decoding

$$L = \frac{\Pr[c_i = 1\,|\mathbf{y}]}{\Pr[c_i = 0\,|\mathbf{y}]}$$

$$\hat{c}_i = \begin{cases} 1 & \text{if } L \geq 1 \\ 0 & \text{if } L < 1 \end{cases}$$

- Constraint

$$\mathbf{c}\mathbf{H}^T = \mathbf{0} \tag{3}$$

Message Passing Algorithm (2/4)

- Messages are probabilities (or likelihood) of “1” or “0” been transmitted
- Similar to *extrinsic information* in BCJR algorithm.
- Two stages of message passing.
  - Probabilities of bit nodes;
  - Probabilities of check nodes.
- Assumptions
  - Independence of *a posterio* probabilities

Message Passing Algorith (3/4)

- Denotations
  - q_{ij} — messages to be passed from bit node c_i to check nodes f_j.
  - r_{ji} — messages to be pased from check node f_j to bit node c_i.
  - $R_j = \{i : h_{ji} = 1\}$ — the set of column locations of the 1's in the jth row
  - $R_{j\backslash i} = \{i' : h_{ji'} = 1\} \setminus \{i\}$ — the set of column locations of the 1's in the jth row, excluding location i.
  - $C_j = \{i : h_{ji} = 1\}$ — the set of row locations of the 1's in the ith column
  - $C_{i\backslash j} = \{i' : h_{j'i} = 1\} \setminus \{j\}$ — the set of row locations of the 1's in the ith column, excluding location j.
  - $p_i = \Pr(c_i = 1|y_i)$

Message Passing Algorith (4/4)

Compute for $\forall i, j$ that satisfies $h_{ij} = 1$.

1. Initialize

$$\begin{aligned} q_{ij}(0) &= 1 - p_i = \Pr(c_i = 0 \,|y_i) = \frac{1}{1 + e^{-2y_i/\sigma^2}} \\ q_{ij}(1) &= p_i = \Pr(c_i = 1 \,|y_i) = \frac{1}{1 + e^{2y_i/\sigma^2}} \end{aligned}$$

2. First half round iteration

$$\begin{aligned} r_{ji}(0) &= \frac{1}{2} + \frac{1}{2} \prod_{i' \in R_{j \setminus i}} \left(1 - 2q_{i'j}(1)\right) \\ r_{ji}(1) &= 1 - r_{ji}(0) \end{aligned}$$

3. Second half round iteration

$$\begin{aligned} q_{ij}(0) &= K_{ij}(1 - p_i) \prod_{j' \in C_{i \setminus j}} r_{j'i}(0) \\ q_{ij}(1) &= K_{ij} p_i \prod_{j' \in C_{i \setminus j}} r_{j'i}(1) \end{aligned}$$

where constants k_{ij} are selected to ensure

$$q_{ij}(0) + q_{ij}(1) = 1$$

4. Soft decision

$$\begin{aligned} Q_i(0) &= K_i (1 - p_i) \prod_{j \in C_i} r_{ij}(0) \\ Q_i(1) &= K_i p_i \prod_{j \in C_i} r_{ij}(1) \end{aligned}$$

where constants k_i are selected to ensure

$$Q_i(0) + Q_i(1) = 1$$

5. Hard decision

$$\hat{c}_i = \begin{cases} 1 & \text{if } Q_i(1) > 0 \\ 0 & \text{elsewhere} \end{cases}$$

If $\hat{\mathbf{c}}\mathbf{H}^T = \mathbf{0}$ or number of iterations exceeds limitation then stop, else go to Step 2.

Log-Domain Algorithm (1/2)

- A log-domain algorithm is desirable because there are many multiplications
  - In log-domain, multiplications will become additions which have less computational complexity;
  - Multiplications may cause overflow or saturation with large numbers of iterations.
- The log-likelihood ratio is defined as

$$L(c_i) \triangleq \log \frac{1-p_i}{p_i}$$

$$L(q_{ij}) \triangleq \log \frac{q_{ij}(0)}{q_{ij}(1)}$$

- The most frequently involved computation can be defined as

$$\phi(x) \triangleq -\log \tanh\left(\frac{1}{2}x\right) = \log \frac{e^x+1}{e^x-1}$$

  - We have the property $\phi^{-1}(x) = \phi(x)$ for $x > 0$

Log-Domain Algorithm (2/2)

- Seperate $L\left(q_{ij}\right)$

$$\begin{aligned} L\left(q_{ij}\right) &= \alpha_{ij}\beta_{ij} \\ \alpha_{ij} &= \operatorname{sign}\left(L\left(q_{ij}\right)\right) \\ \beta_{ij} &= \operatorname{abs}\left(L\left(q_{ij}\right)\right) \end{aligned}$$

- The log-domain algorithm
  1. Initialize

$$L\left(q_{ij}\right) = 2y_i/\sigma^2$$

  2. First half round iteration

$$L\left(r_{ji}\right) = \prod_{i' \in R_{j\setminus i}} \alpha_{i'j} \cdot \phi\left[\sum_{i' \in R_{j\setminus i}} \phi\left(\beta_{i'j}\right)\right]$$

3. Second half round iteration

$$L\left(q_{ij}\right) = L\left(c_i\right) + \sum_{j' \in C_{i \backslash j}} L\left(r_{j'i}\right)$$

4. Soft decision

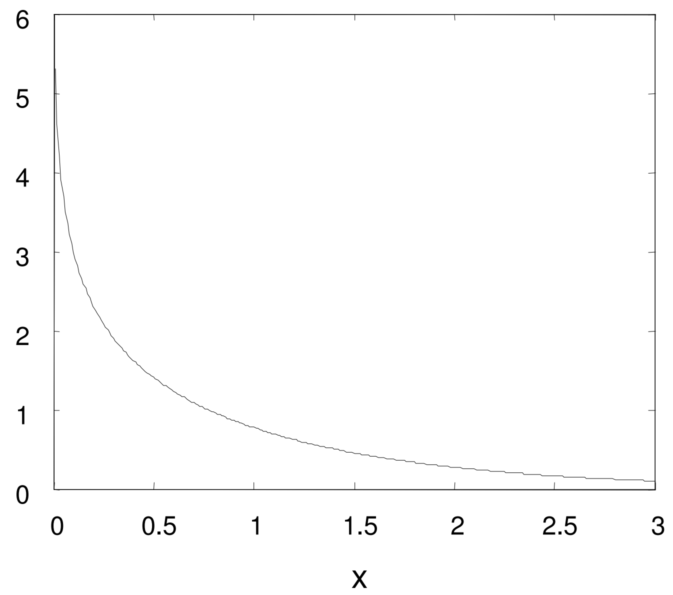
$$L\left(Q_i\right) = L\left(C_i\right) + \sum_{j \in C_i} L\left(r_{ji}\right)$$

5. Hard decision

$$\hat{c}_i = \begin{cases} 1 & \text{if } L\left(Q_i\right) < 0 \\ 0 & \text{elsewhere} \end{cases}$$

If $\hat{\mathbf{c}}\mathbf{H}^T = \mathbf{0}$ or number of iterations exceeds limitation then stop, else go to Step 2.

Min-Sum Algorithm



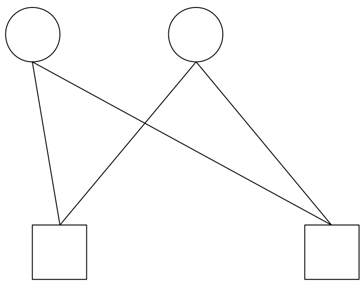
- The shape of $\phi(x)$.
- The smallest β_{ij} dominates.

$$\begin{aligned}\phi\left[\sum_{i'}\phi\left(\beta_{i'j}\right)\right] &\approx \phi\left[\phi\left(\min_{i'}\beta_{i'j}\right)\right]\\ &= \min_{i'}\beta_{i'j}\end{aligned}$$

- The min-sum algorithm is the log-domain algorithm with step 2 modified by

$$L\left(r_{ji}\right) = \prod_{i'\in R_{j\backslash i}}\alpha_{i'j}\cdot\min_{i'\in R_{j\backslash i}}\beta_{i'j}$$

Design of LDPC Codes



- Large d_{min}
- No short cycles
  - Cycles exist in Tanner graphs.
  - Cycles hurt the performance of the message passing algorithm because they invalidate the assumption of independence.
  - The shortest possible cycle has the length 4.
  - Although we can eliminate all cycles with length 4, we may still have cycles with length 6.
- No eliminating sets
  - Applications in binary erasure channels.

Open Problems in LDPC Codes

- LDPC codes with near-capacity performance
  - Very long codewords, many iterations, low signal-to-noise ratio.
- LDPC codes with relatively short codeword
  - High coding rate $r \approx 1$
  - Short codewords enable easy encoding
- Combination with other technologies
  - LDPC codes with OFDM systems
  - LDPC codes with MIMO systems

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