An Introduction to Low Density Parity Check (LDPC) Codes

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Outline

1. History of LDPC codes
2. Properties of LDPC codes
3. Basics of LDPC codes
   • Encoding of LDPC codes
   • Iterative decoding of LDPC codes
   • Simplified approximations of LDPC decoders
4. Applications of LDPC codes
Features of LDPC Codes

- Approaching Shannon capacity
  - For example, 0.3 dB from Shannon limit
  - Irregular LDPC code with code length 1 million. (Richardson: 1999)
  - An closer design from (Chung: 2001), 0.0045 dB away from capacity

- Good block error correcting performance

- Low error floor
  - The minimum distance is proportional to code length

- Linear decoding complexity in time

- Suitable for parallel implementation
History of LDPC Codes

- Invented by Robert Gallager in his 1960 MIT Ph. D. dissertation. Long time being ignored due to
  1. Requirement of high complexity computation
  2. Introduction of Reed-Solomon codes
  3. The concatenated RS and convolutional codes were considered perfectly suitable for error control coding.

Foundamentals of Linear Block Codes

- The structure of a code is completely described by the generator matrix $G$ or the parity check matrix $H$.
- The capacity of correcting symbol errors in a codeword is determined by the minimum distance ($d_{\text{min}}$).
  - $d_{\text{min}}$ is the least weight of the rows in $G$.
  - $d_{\text{min}}$ is the least number of columns in $H$ that sum up to 0.
  - Example: (7, 4) Hamming code

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$
Properties of LDPC Codes

- H is sparse.
  - Very few 1’s in each row and column.
  - Expected large minimum distance.

- Regular LDPC codes
  - H contains exactly $W_c$ 1’s per column and exactly $W_r = W_c(n/m)$ 1’s per row, where $W_c \ll m$.
  - The above definition implies that $W_r \ll n$.
  - $W_c \geq 3$ is necessary for good codes.

- If the number of 1’s per column or row is not constant, the code is an irregular LDPC code.
  - Usually irregular LDPC codes outperform regular LDPC codes.
A Sample LDPC Code

\[ H = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\vdots
\end{bmatrix} \]

- \( W_c = 3; \)
- Any two columns have an overlap of at most one 1;
- The sparse property allows us to avoid overlapping;
- In the part of \( H \) shown, there does not exist a set of columns that add up to 0.
- The above facts make the \( d_{\min} \) large;
- \( G \) is found by Gaussian elimination.
  - \( H \) can be put in the form \( H = \begin{bmatrix} P^T : I \end{bmatrix} \).
  - The generator matrix \( G = \begin{bmatrix} I : P \end{bmatrix} \).
Encoding of LDPC Codes

- General encoding of systematic linear block codes
  \[ c = xG = \left[ x : xP \right] \] (1)

- Issues with LDPC codes
  - The size of \( G \) is very large.
  - \( G \) is not generally sparse.
  - Example: A (10000, 5000) LDPC code.
    * \( P \) is 5000 \( \times \) 5000.
    * We may assume that the density of 1’s in \( P \) is 0.5
    * There are \( 12.5 \times 10^6 \) 1’s in \( P \)
    * \( 12.5 \times 10^6 \) addition (XOR) operations are required to encode one codeword.

- An alternative approach to simplified encoding is to design the LDPC code via algebraic or geometric methods.
  - Such “structured” codes can be encoded with shift register circuits.
Iterative Decoding of LDPC Codes

- General decoding of linear block codes
  - Only if $c$ is a valid codeword, we have
    \[ cH^T = 0 \] (2)
  - For binary symmetric channel (BSC), the received codeword is $c$ added with an error vector $e$.
  - The decoder needs to find out $e$ and flip the corresponding bits.
  - The decoding algorithm is based on linear algebra.

- Graph-based algorithms
  - Sum-product algorithm for general graph-based codes;
  - MAP (BCJR) algorithm for trellis graph-based codes;
  - Message passing algorithm for bipartite graph-based codes.
Tanner Graph

- Bipartite graph
  A bipartite graph is an undirected graph whose nodes may be separated into two classes, where edges only connect two nodes not residing in the same class.

- Tanner graph
  The two classes of nodes in a Tanner graph are the *bit nodes* and the *check nodes*.

- Example: An (8, 4) product code

\[
H = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Message Passing Algorithm (1/4)

- Maximum likelihood (ML) decoding

$$L = \frac{\Pr[c_i = 1 | y]}{\Pr[c_i = 0 | y]}$$

$$\hat{c}_i = \begin{cases} 
1 & \text{if } L \geq 1 \\
0 & \text{if } L < 1 
\end{cases}$$

- Constraint

$$cH^T = 0 \quad (3)$$
Message Passing Algorithm (2/4)

- Messages are probabilities (or likelihood) of “1” or “0” been transmitted
- Similar to *extrinsic information* in BCJR algorithm.
- Two stages of message passing.
  - Probabilities of bit nodes;
  - Probabilities of check nodes.
- Assumptions
  - Independence of *a posteriori* probabilities
Message Passing Algorithm (3/4)

- Denotations
  - $q_{ij}$ — messages to be passed from bit node $c_i$ to check nodes $f_j$.
  - $r_{ji}$ — messages to be passed from check node $f_j$ to bit node $c_i$.
  - $R_j = \{i : h_{ji} = 1\}$ — the set of column locations of the 1’s in the $j$th row
  - $R_j \setminus i = \{i' : h_{ji'} = 1\} \setminus \{i\}$ — the set of column locations of the 1’s in the $j$th row, excluding location $i$.
  - $C_j = \{i : h_{ji} = 1\}$ — the set of row locations of the 1’s in the $i$th column
  - $C_i \setminus j = \{i' : h_{j'i} = 1\} \setminus \{j\}$ — the set of row locations of the 1’s in the $i$th column, excluding location $j$.
  - $p_i = \Pr(c_i = 1|y_i)$
Message Passing Algorithm (4/4)

Compute for $\forall i, j$ that satisfies $h_{ij} = 1$.

1. Initialize

$$q_{ij} (0) = 1 - p_i = \Pr (c_i = 0 | y_i) = \frac{1}{1 + e^{-2y_i/\sigma^2}}$$

$$q_{ij} (1) = p_i = \Pr (c_i = 1 | y_i) = \frac{1}{1 + e^{2y_i/\sigma^2}}$$

2. First half round iteration

$$r_{ji} (0) = \frac{1}{2} + \frac{1}{2} \prod_{i' \in R_j \backslash i} (1 - 2q_{i'j} (1))$$

$$r_{ji} (1) = 1 - r_{ji} (0)$$

3. Second half round iteration

$$q_{ij} (0) = K_{ij} (1 - p_i) \prod_{j' \in C_i \backslash j} r_{j'i} (0)$$

$$q_{ij} (1) = K_{ij} p_i \prod_{j' \in C_i \backslash j} r_{j'i} (1)$$
where constants $k_{ij}$ are selected to ensure

$$q_{ij}(0) + q_{ij}(1) = 1$$

4. Soft decision

$$Q_i(0) = K_i (1 - p_i) \prod_{j \in C_i} r_{ij}(0)$$

$$Q_i(1) = K_i p_i \prod_{j \in C_i} r_{ij}(1)$$

where constants $k_i$ are selected to ensure

$$Q_i(0) + Q_i(1) = 1$$

5. Hard decision

$$\hat{c}_i = \begin{cases} 1 & \text{if } Q_i(1) > 0 \\ 0 & \text{elsewhere} \end{cases}$$

If $\hat{c}H^T = 0$ or number of iterations exceeds limitation then stop, else go to Step 2.
Log-Domain Algorithm (1/2)

- A log-domain algorithm is desirable because there are many multiplications
  - In log-domain, multiplications will become additions which have less computational complexity;
  - Multiplications may cause overflow or saturation with large numbers of iterations.
- The log-likelihood ratio is defined as
  \[ L(c_i) \triangleq \log \frac{1 - p_i}{p_i} \]
  \[ L(q_{ij}) \triangleq \log \frac{q_{ij}(0)}{q_{ij}(1)} \]
- The most frequently involved computation can be defined as
  \[ \phi(x) \triangleq - \log \tanh \left( \frac{1}{2} x \right) = \log \frac{e^x + 1}{e^x - 1} \]
  - We have the property \( \phi^{-1}(x) = \phi(x) \) for \( x > 0 \)
Log-Domain Algorithm (2/2)

- Separate $L(q_{ij})$

\[ L(q_{ij}) = \alpha_{ij} \beta_{ij} \]
\[ \alpha_{ij} = \text{sign}(L(q_{ij})) \]
\[ \beta_{ij} = \text{abs}(L(q_{ij})) \]

- The log-domain algorithm
  1. Initialize

\[ L(q_{ij}) = 2y_i/\sigma^2 \]

  2. First half round iteration

\[ L(r_{ji}) = \prod_{i' \in R_{j \setminus i}} \alpha_{i'j} \cdot \phi \left[ \sum_{i' \in R_{j \setminus i}} \phi(\beta_{i'j}) \right] \]
3. Second half round iteration

\[ L(q_{ij}) = L(c_i) + \sum_{j' \in C_i \setminus j} L(r_{j'i}) \]

4. Soft decision

\[ L(Q_i) = L(C_i) + \sum_{j \in C_i} L(r_{ji}) \]

5. Hard decision

\[ \hat{c}_i = \begin{cases} 
1 & \text{if } L(Q_i) < 0 \\
0 & \text{elsewhere} 
\end{cases} \]

If \( \hat{c}H^T = 0 \) or number of iterations exceeds limitation then stop, else go to Step 2.
Min-Sum Algorithm

- The shape of $\phi(x)$.
- The smallest $\beta_{i,j}$ dominates.

\[
\phi \left[ \sum_{i'} \phi (\beta_{i'j}) \right] \approx \phi \left[ \phi \left( \min_{i'} \beta_{i'j} \right) \right] = \min_{i'} \beta_{i'j}
\]

- The min-sum algorithm is the log-domain algorithm with step 2 modified by

\[
L(r_{ji}) = \prod_{i' \in R_j \setminus i} \alpha_{i'j} \cdot \min_{i' \in R_j \setminus i} \beta_{i'j}
\]
Design of LDPC Codes

- Large $d_{\text{min}}$
- No short cycles
  - Cycles exist in Tanner graphs.
  - Cycles hurt the performance of the message passing algorithm because they invalidate the assumption of independence.
  - The shortest possible cycle has the length 4.
  - Although we can eliminate all cycles with length 4, we may still have cycles with length 6.
- No eliminating sets
  - Applications in binary erasure channels.
Open Problems in LDPC Codes

- LDPC codes with near-capacity performance
  - Very long codewords, many iterations, low signal-to-noise ratio.

- LDPC codes with relatively short codeword
  - High coding rate $r \approx 1$
  - Short codewords enable easy encoding

- Combination with other technologies
  - LDPC codes with OFDM systems
  - LDPC codes with MIMO systems
References


